Review Packet
Pre-Calculus
2015-2016

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This packet includes all algebra and geometry skills that are necessary for Pre-Calculus and AP Calculus. It is preferred that you are proficient in all of these areas.

Table of Contents

A. Algebra Review

a. Classifying Numbers

- Integers - whole numbers including negative numbers and zero $(1,2,3,4,\ldots)$
- Rational Numbers - a number that can be expressed as a quotient $(\frac{1}{2},\frac{1}{3},\ldots)$
- Numerator - The top of the fraction
- Denominator - The bottom of the fraction, which cannot be 0
- Irrational Numbers - a decimal number that neither repeats or ends $(3.1452215\ldots)$
- Real Numbers - the set of rational and irrational numbers combined
- Counting Numbers - also called natural numbers; used to count things (1,2,3…)

B. Integer Exponents

a. Evaluating Expressions with Exponents

- $a^2 = a \cdot a$ "a number multiplied to itself"
- $a^6 = a \cdot a \cdot a \cdot a \cdot a \cdot a$ "a number multiplied to itself 6 times"
- $(-2)^2 = (-2)(-2) = 4$ "a negative number squared is positive"
- $-(2)^2 = -(2)(2) = -4$ "number remains negative if the negative is outside the parenthesis"
- $a^{-2} = \frac{1}{a^2}$ "negative exponents can move the term from the numerator to the denominator, and become positive. Or vice-versa"
- $\frac{2}{x^{-1}} = 2x$
- $a^0 = 1$ "anything raised to the zero power is 1"

b. Work with Law of Exponents

- $a^m a^n = a^{m+n}$ "you may add the exponents when multiplying similar bases (both a base of 'a')"
- $(a^m)^n = a^{mn}$ "exponents with exponents may be multiplied"
- $(ab)^n = a^n b^n$ "an exponent outside of the parenthesis of a monomial (no plus or minus) may be applied to each term inside"
- $(\frac{a}{b})^n = \frac{a^n}{b^n}$
- $(\frac{a}{b})^{-n} = \frac{b^n}{a^n}$ "a negative exponent may flip a fraction"
- $\frac{a^m}{a^n} = a^{m-n}$ "you may subtract the exponents when dividing similar bases (both a base of 'a')"

c. Use a calculator to evaluate exponents

Example:

- Evaluate $(-4.3)^4$
  On your calculator, type: (-4.3)^(4)
- Evaluate $(2)^{-4}$
  On your calculator, type: (2)^(-4)

d. Scientific Notation

- This can be used to express very large or very small numbers in a simpler manner. It must be a number between 1 and 10.
- $4{,}328{,}201$ can be written as 4.3×10^6
- 0.00035 can be written as 3.5×10^{-4}

C. Polynomials

a. Recognizing Monomials

- The form looks like: ax^k where a is a constant and the variable x is raised to a power. The a is considered the coefficient of the monomial. The k is the degree of the monomial.
- Example: $3x^2$

b. Recognizing Polynomials

- A binomial looks like this: $x^2 - 3$, the addition or subtraction separates it into two terms.
- A trinomial looks like this: $2x^2 + 4x - 8$, the addition and subtraction separates it into three terms.

c. Adding and Subtracting Polynomials

- You can add/subtract the coefficients if the variable (and it's exponent) are the same. I refer to this as "combining like terms".
- Example:
  - $(8x^3 - 2x^2 + 6x - 2) + (3x^4 - 2x^3 + x + 1)$
  - $[3x^4] + [8x^3 + (-2x^3)] + [(-2x^2)] + [6x + x] + [(-2) + 1]$
    $3x^4 + 6x^3 - 2x^2 + 7x - 1$

d. Multiplying Polynomials

- Horizontal Multiplication
  - $(3x + 4)(x^2 - x + 2)$
    $3x(x^2 - x + 2) + 4(x^2 - x + 2)$ "distribute the $3x$ through the trinomial, as well as the 4"

  $[3x^3 - 3x^2 + 6x] + [4x^2 - 4x + 8]$ Then combine like terms (see above letter 'c' add/subtract polynomials)

e. Special Products

- When two binomials are being multiplied, use **FOIL**
  First **O**utside **I**nside **L**ast
  - Example: $(x + 2)(3x - 1)$
    First : $(x)(3x) = 3x^2$
    Outside: $(x)(-1) = -x$
    Inside: $(2)(3x) = 6x$
    Last: $(2)(-1) = -2$
    All combined gives you: $3x^2 + 5x - 2$

D. Factoring Polynomials

Example: $4x + 8 = 4(x + 2)$

The 4 was a common factor of $4x$ and 8, so divide it out and put it in front.

a. Difference of Two Squares

- $x^2 - a^2 = (x + a)(x - a)$
  - Example:

$x^2 - 16 = (x+4)(x-4)$

b. Factor a second-degree polynomial: $x^2 + Bx + C$

- Find the factors of C whose sum is B
- Example: $x^2 + 6x + 8$

  Factors of $8 = 4 \cdot 2\,, or\ 8 \cdot 1$

  Which pair adds up to 6? …. …. **4 and 2**

  Result: $(x+4)(x+2)$

c. Factor by grouping

- Example: factor completely by grouping, $x^3 - 4x^2 + 2x - 8$

  Pair up: $(x^3 - 4x^2) + (2x - 8)$

  Factor each: $x^2(x-4) + 2(x-4)$

  Factor out the "$x-4$": $(x-4)(x^2+2)$

d. Factor a second-degree polynomial: $Ax^2 + Bx + C$

- Similar to previous method, guess (with factors of C) and check by FOILing the binomial back to the trinomial.
- Bottoms-Up Method

E. Solving Equations

a. Any equation with one variable can be solved (isolated on one side of the equal sign). The solutions are sometimes referred to as roots, or zeros.

- You must do the OPPOSITE order of operations.
- Order of operations: PEMDAS
  Parentheses/Exponents - Multiplication/Division - Addition/Subtraction
- Example: $3x - 5 = 4$

  $3x - 5\ + 5 = 4 + 5$ add 5 to both sides

  $3x = 9$

  $\frac{3x}{3} = \frac{9}{3}$ divide both sides by 3

  $x = 3$ 3 is your solution!

F. Rational Expressions: "Fractions"

a. Find the domain and range of a function

- The denominator of a fraction cannot equal zero, because dividing by zero is undefined. Anytime you have a fraction, check what values of the variable make the denominator zero.
- Example: $y = \frac{5}{x-2}$

  If $x = 2$, then the denominator is zero. Therefore, this fraction's domain is everything BUT $x = 2$. Or, you can say the domain is $\{x | x \neq 2\}$which means "all values of x but 2". The range (possible values of y) may be anything (negative or positive) but it cannot be zero. Reason: it's impossible to have a result of zero when your function is "5 divided by something". So you may say the range is anything but zero, or $\{y | y \neq 0\}$.

- Example: $y = \sqrt{9 - x^2}$

  With square root functions, you need only to worry about what values of x make the inside of the radical negative (because you can't square root a negative number!). Consider if x is greater than 3, then we will have 9 minus a bigger number than 9, which will result in a negative inside the radical. We only want values of x that work. So, you may say that the domain is $x < 3$. The range (possible values of y) may be anything but a negative number. Reason: it is impossible for a square root to result in a negative number. So, you may say that the range is $y \geq 0$.

b. Reduce a rational expression to lowest terms

- Factor out all common terms and cancel out terms that appear on the top and bottom of the fraction. **FACTOR FIRST. THEN CANCEL!**
- Example: $\frac{x^2-9}{x^2+4x+3}$ factor numerator and denominator

  $\frac{(x+3)(x-3)}{(x+3)(x+1)}$ similar quantities of $(x+3)$ on top and bottom

  $\frac{(x-3)}{(x+1)}$ simplified rational expression
- Example: $\frac{4x^2-2x}{4x}$ factor the numerator

  $\frac{2x(2x-1)}{4x}$ after factoring, you may cancel out/simplify terms

  $\frac{(2x-1)}{2}$ simplified rational expression

c. Multiply and divide rational expressions

- When you **multiply fractions**, you multiply the numerators together and you multiply the denominators together (just like multiplying polynomials).
- Example: $\frac{2+x}{6x} \cdot \frac{7x^2}{1+x} = \frac{(2+x)(7x^2)}{(6x)(1+x)} = \frac{14x^2+7x^3}{6x+6x^2}$
- When you **divide fractions**, you flip the second fraction and change the operation from division to multiplication.
- Example: $\frac{2+x}{6x} \div \frac{7x^2}{1+x} = \frac{2+x}{6x} \cdot \frac{1+x}{7x^2}$ the second fraction was flipped and is multiplied.

d. Add and subtract rational expressions

- You must make common denominators first! Then, add/subtract the numerators just like you would normally with polynomials. Denominator remains the same.
- Example: $\frac{3x}{x+1} - \frac{x-2}{x}$ we must make the denominators the same before subtracting the numerators.

  $\frac{(3x)(x)}{(x+1)(x)} - \frac{(x-2)(x+1)}{(x)(x+1)}$ multiply to the top and bottom of each fraction to make the denominators the exact same

  $\frac{3x^2}{(x+1)(x)} - \frac{x^2-x-2}{(x+1)(x)}$ simplify numerators. denominators remain the same (do not multiply or attempt to simplify).

  $\frac{(3x^2)-(x^2-x-2)}{(x+1)(x)}$ subtract numerators. In this case, you'll need to distribute the subtraction to each term in trinomial.

  $\frac{2x^2+x+2}{(x+1)(x)}$ final result

e. Simplify mixed quotients

- You must simplify mixed quotients to solve for a variable or to analyze.
- Mixed quotients are expressions with fractions inside of fractions.

- Example: $\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}}$ You must first combine fractions in the numerator ($\frac{1}{2}+\frac{3}{x}$) so that the entire numerator is one fraction.

$$\frac{\frac{1(x)}{2(x)}+\frac{3(2)}{x(2)}}{\frac{x+3}{4}}$$

Add the top fractions like you would normally (making common denominators).

$$\frac{\frac{x}{2x}+\frac{6}{2x}}{\frac{x+3}{4}}$$

Add the numerator fractions together.

$$\frac{\frac{6+x}{2x}}{\frac{x+3}{4}}$$

Now that you have a simple quotient with two fractions, flip the bottom fraction and multiply like you would normally with two divided fractions.

$$\frac{6+x}{2x}\cdot\frac{4}{x+3}$$

Multiply fractions. (see multiply rational expressions)

G. Radicals

a. work with square roots and any other roots

- Properties of Radicals:
  - $\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$
  - $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
  - $\sqrt[n]{a^m}=(\sqrt[n]{a})^m$
  - $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$
- Other properties
  - $\sqrt{x^2}=|x|$
  - $\sqrt{x}\cdot\sqrt{x}=x$ "$x^{1/2}\cdot x^{1/2}=x^{1/2+1/2}=x^1=or\ just\ x$"
  - $\sqrt{x}=x^{1/2}$
  - $\sqrt[3]{x^2}=x^{2/3}$

b. Simplify Radicals

- Look to factor radicals into products that involve perfect square numbers.
- Example: $\sqrt{32}$

$\sqrt{16}\cdot\sqrt{2}$ 16 is a perfect square number ($\sqrt{16}=4$)

$4\cdot\sqrt{2}$

$4\sqrt{2}$ final result

c. Rationalizing Denominators

- Radicals cannot exist in the denominators of fractions, so we ‘rationalize’ the fraction so that the radical only exists in the numerator.
- Example: $\frac{4}{\sqrt{2}}$ multiply the top and bottom by the radical

  $\frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ now multiply the top and multiply the bottom

  $\frac{4\sqrt{2}}{\sqrt{4}}$ the denominator will be a perfect square

  $\frac{4\sqrt{2}}{2}$ simplify the fraction

  $\frac{2\sqrt{2}}{1} = 2\sqrt{2}$ final result
- Example: $\frac{\sqrt{x}-2}{\sqrt{x}+2}$

  $\frac{\sqrt{x}-2}{\sqrt{x}+2} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2}$ multiply the top and the bottom by $\sqrt{x}-2$ which is simply the denominator of the fraction with the opposite operation.

  $\frac{(\sqrt{x}-2)(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}$ FOIL the numerator and FOIL the denominator, because they are multiplied binomials. (see multiplying polynomials)

  $\frac{x-2\sqrt{x}-2\sqrt{x}+4}{x-2\sqrt{x}+2\sqrt{x}-4}$ Combine like terms

  $\frac{x-4\sqrt{x}+4}{x-4}$ final simplified result

d. Simplify expressions with rational exponents

- Radicals may be treated as rational exponents, so that they would follow all the same exponential properties as any other exponents.
- Example: $(\frac{2x^{1/3}}{y^{2/3}})^{-3}$

  $(\frac{y^{2/3}}{2x^{1/3}})^3$ The negative 3 exponent flips the fraction. The exponent now is positive after the flip.

  $\frac{(y^{2/3})^3}{(2x^{1/3})^3}$ Apply the exponent to the numerator and the denominator.

  $\frac{y^{\frac{2}{3}\cdot\frac{3}{1}}}{8x^{\frac{1}{3}\cdot\frac{3}{1}}}$ You will multiply the exponents (laws of exp.) In this situation, the exponents are fractions. Multiply like fractions.

  $\frac{y^2}{8x}$ Final simplified result
- Example: $(x^2+1)^{1/2} + x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$ Simplify into a single quotient

  $(x^2+1)^{1/2} + x \cdot (x^2+1)^{-1/2} \cdot x$ The $1/2$ and the 2 cancel out

  $(x^2+1)^{1/2} + x^2(x^2+1)^{-1/2}$ The x’s can be multiplied: x^2

  $(x^2+1)^{1/2} + \frac{x^2}{(x^2+1)^{1/2}}$ Negative exponent (½) becomes +

$\frac{(x^2+1)^{1/2}}{1} + \frac{x^2}{(x^2+1)^{1/2}}$ Rewrite as sum of two fractions

$\frac{(x^2+1)^{1/2}}{1} \cdot \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}} + \frac{x^2}{(x^2+1)^{1/2}}$ Make common denominators

$\frac{(x^2+1)}{(x^2+1)^{1/2}} + \frac{x^2}{(x^2+1)^{1/2}}$ Combine (add) fractions

$\frac{x^2+1+x^2}{(x^2+1)^{1/2}}$ Combine like terms on top

$\frac{2x^2+1}{(x^2+1)^{1/2}}$ Final simplified result

H. Geometry Review

a. Pythagorean Theorem

- $a^2 + b^2 = c^2$
- a and b represent the two legs of the right triangle
- c represents the hypotenuse of the right triangle

b. Geometry Formulas

- Rectangle $l = length, w = width$
  - $Area = lw$ $Perimeter = 2l + 2w$
- Triangle $b = base, h = height$
  - $Area = \frac{1}{2}bh$
- Circle $r = radius, d = diameter$
  - $Area = \pi r^2$ $Circumference = 2\pi r$ or πd
- Rectangular Box $l = length, w = width, h = height$
  - $Volume = lwh$
- Sphere $r = radius$
  - $Volume = \frac{4}{3}\pi r^3$ $Surface\ Area = 4\pi r^2$
- Right Circular Cylinder $r = radius, h = height$
  - $Volume = \pi r^2 h$

I. Completing the Square

- This comes in handy when you need to change a second-degree polynomial into a perfect square binomial quantity.
- Example and Method

  $x^2 + 6x = 3$

  $x^2 + 6x + (9) = 3 + (9)$ Add 9 to both sides. 9 was found from taking half of the coefficient from the "middle" $6x$ term, then squaring it.

  $(x^2 + 6x + 9) = 12$ The trinomial can be factored into two binomials.

  $(x + 3)^2 = 12$ The resulting binomial will be $x +$___ whatever half of that "middle" term was.
- Another example

  $x^2 - 8x = 0$

  $x^2 - 8x + (16) = 0 + (16)$

$(x-4)^2 = 16$ In this case, the binomial is subtraction because of the "middle" term being a $-8x$.

- Multivariable example

  $x^2 + y^2 - 2x + 4y - 4 = 0$

  $(x^2 - 2x) + (y^2 + 4y) = 4$ Group similar variables together. Add 4 to other side of equal sign.

  $[x^2 - 2x + (1)] + [y^2 + 4y + (4)] = 4 + (1) + (4)$

  $(x-1)^2 + (y+2)^2 = 9$ Each variable has it's own square binomial

J. Solving for the zeros (or roots)

a. Factoring

- Zeros for a trinomial may be found by factoring it into a product of two binomials equaling zero.
- Example: $x^2 + 7x + 6 = 0$

$(x+6)(x+1) = 0$	Factor
$(x+6) = 0 \quad and \quad (x+1) = 0$	Separate
$x = -6 \quad and \quad x = -1$	Two solutions

b. Quadratic Formula

- If factoring doesn't work, the quadratic formula will work.
- Formula:
  - $ax^2 + bx + c = 0$

    $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Example: $x^2 + 7x + 6 = 0$

  $a = 1, b = 7, c = 6$

$\frac{-(7) \pm \sqrt{(7)^2 - 4(1)(6)}}{2(1)}$	Plug in values of a,b, & c
$\frac{-7 \pm \sqrt{49 - 24}}{2}$	Simplify
$\frac{-7 \pm \sqrt{25}}{2}$	Simplify
$\frac{-7 \pm 5}{2}$	Consider the expression as an addition and as a subtraction
$\frac{-7+5}{2}$ and $\frac{-7-5}{2}$	Evaluate each one to find both solutions
$x = \frac{-2}{2} = -1$ and $x = \frac{-12}{2} = -6$	Two solutions

c. Graphing Calculator

- You can use your graphing calculator to solve complex polynomials.
- Steps:
  - Type the function into y_1 and graph
  - You may need to adjust viewing window to see the graph
  - Hit "2nd" , "TRACE" to acquire the "CALC" option
  - Choose "ZERO" option
  - Use arrows to move cursor towards one of the x-intercepts
  - "lower" or "left" bound needs to be found first, so move cursor

directly beside x-intercept to the left. Press “ENTER”
    - Click arrow key once to the right to capture the “upper” or “right” bound. Press “ENTER”
    - The Calculator will read “GUESS?”. Press “ENTER” again
    - The zero will be displayed as an x and y coordinate. The y coordinate will read “0”. The x coordinate is your zero (root).

K. Equations of Lines

a. Point-Slope and Slope-Intercept Form

- It is important to know each form because one may be easier and more convenient than the other according to the situation.
- **Point-Slope Form**
    - $(y - y_1) = m(x - x_1)$
    - y_1 is the y-coordinate, x_1 is the x-coordinate, and m is the slope
- Example: Write the equation of the line that passes through the point $(1, -2)$ and has a slope of $3/4$.

$[y-(-2)] = \frac{3}{4}[x-(1)]$	Plug in $1$for $x_1$and -2 for y_1
$(y+2) = \frac{3}{4}(x-1)$	
$y+2 = \frac{3}{4}x - \frac{3}{4}$	Distribute $\frac{3}{4}$ through parentheses
$y = \frac{3}{4}x - \frac{3}{4} - 2$	Subtract 2 to other side to solve for y
$y = \frac{3}{4}x - \frac{11}{4}$	Final result is in $y = mx + b$ form

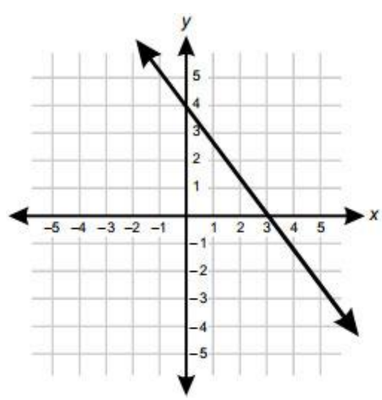
- **Slope-Intercept Form**
    - $y = mx + b$
- Example: Write the equation of the line that has a slope of 3 and a y-intercept of 7.

$y = (3)x + (7)$	Plug in 3 for m and 7 for b.
$y = 3x + 7$	Final result

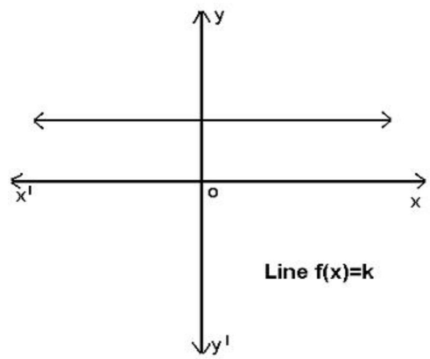
L. Functions

a. Library of functions

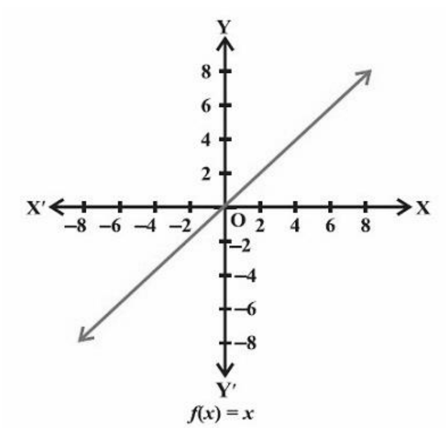
- **Linear Functions** $f(x) = mx + b$ “m is slope. b is y-intercept”



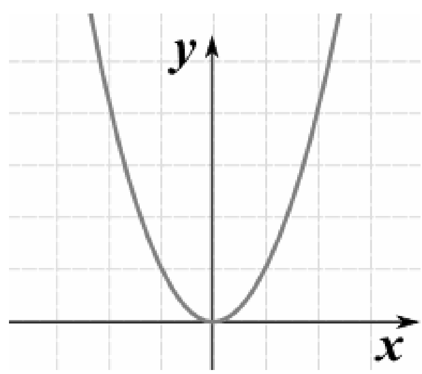
- **Constant Function** $f(x) = b$ "horizontal line at $y = b$"



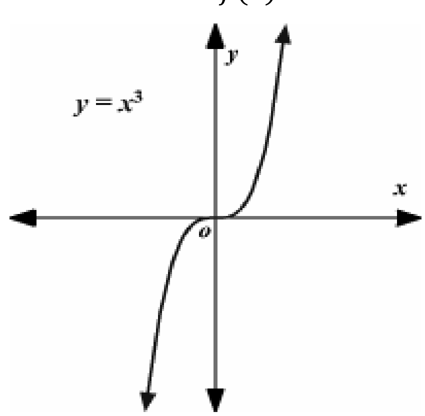
- **Identity Function** $f(x) = x$ "diagonal line through origin"



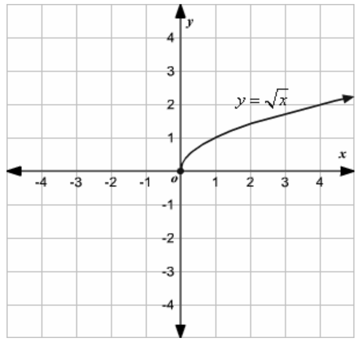
- **Square Function** $f(x) = x^2$ **"**Parabola"



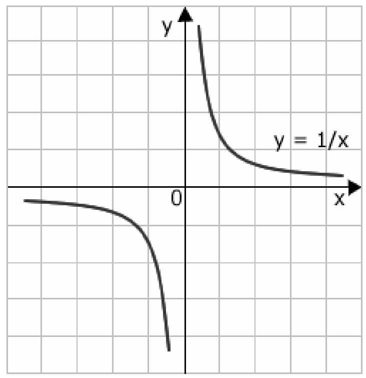
- **Cube Function** $f(x) = x^3$ "S-like curve"



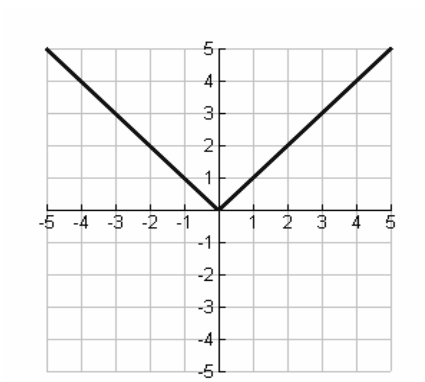
- **Square Root Function** $f(x) = \sqrt{x}$



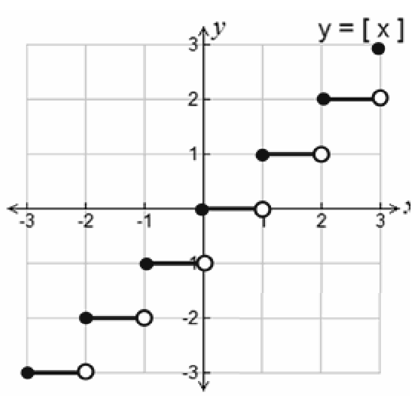
- **Reciprocal Function** $f(x) = \frac{1}{x}$



- **Absolute Value Function** $f(x) = |x|$ "looks like a V"



- **Greatest Integer Function** $f(x) = int\ (x)$ "stair-steps"



b. Even and Odd functions

- A function is even when $f(-x) = f(x)$, or in other words, a function will remain the same EVEN if a negative value or variable is plugged in.
- Example: $f(x) = x^2$
  Plug in "$-x$"into the function: $(-x)^2$ and it simplifies back to x^2.
- A function is odd when $f(-x) = -f(x)$, or in other words, a function will ODDLY change signs when a negative value or variable is plugged in.
- Example: $f(x) = x^3$
  Plug in "$-x$" into the function: $(-x)^3$ and it simplifies to $-x^3$.

c. Operations between functions

- Two functions may be added, subtracted, multiplied, or divided
- Example: $f(x) = 3x$ and $g(x) = x + 1$
  $f(x) + g(x) = [3x] + [x + 1] = 4x + 1$
  $f(x) \cdot g(x) = [3x] \cdot [x + 1] = 3x^2 + 3x$
  $f(x) \div g(x)$ or $\frac{f(x)}{g(x)} = \frac{3x}{x+1}$

d. Composite functions

- Composite functions are functions that are "plugged" into one another
- Example: $f(x) = 2x$ and $g(x) = 3x - 1$
  $f(g(x)) = 2(3x - 1) = 6x - 2$ "f of g of x. Or, g function plugged into the f function."
  $g(f(x)) = 3(2x) - 1 = 6x - 1$ "g of f of x. Not necessarily the same when switched. $f(g(x)) \neq g(f(x))$

e. Piecewise-defined functions

- We may have a function made up entirely by pieces of other functions
- Example:

$$f(x) = \begin{cases} x^2, & x < 0 \\ 2, & 0 \le x \le 3 \\ 4 - x, & x > 3 \end{cases}$$